Access to Science, Engineering and Agriculture: Mathematics 1 MATH00030 Semester 1 2015-2016 Exam Solutions

All the exam questions are unseen.

1. (a) (i) $\dfrac{2}{9} - \dfrac{4}{5} = \dfrac{(2)(5) + (-4)(9)}{(9)(5)} = \dfrac{-26}{45} = -\dfrac{26}{45}$.

(ii) $\dfrac{7}{5} \times \left(-\dfrac{6}{11}\right) = \dfrac{(7)(-6)}{(5)(11)} = \dfrac{-42}{55} = -\dfrac{42}{55}$.

(iii) $\dfrac{4}{5} \div \dfrac{5}{4} = \dfrac{4}{5} \times \dfrac{4}{5} = \dfrac{(4)(4)}{(5)(5)} = \dfrac{16}{25}$.

(iv) $-3^2 = -\left(3^2\right) = -9$.

(v) $\left(\dfrac{9}{4}\right)^{-\frac{3}{2}} = \dfrac{1}{\left(\dfrac{9}{4}\right)^{\frac{3}{2}}} = \dfrac{1}{\left(\left(\dfrac{9}{4}\right)^{\frac{1}{2}}\right)^3} = \dfrac{1}{\left(\dfrac{3}{2}\right)^3} = \dfrac{1}{27/8} = \dfrac{8}{27}$.

(vi) $7 + 7 \div (-6) + 3 = 7 + \left(-\dfrac{7}{6}\right) + 3 = \dfrac{(7)(6) - 7 + (3)(6)}{6} = \dfrac{53}{6}$.

(vii) Since $3^3 = 27$, it follows that $\log_3 27 = 3$.

(viii) Since $2^{-2} = \dfrac{1}{4}$, it follows that $\log_2 \dfrac{1}{4} = -2$. **[8]**

(b) (i) $x^3 \times x^{-1} = x^{3+(-1)} = x^2$.

(ii) $x^{\frac{1}{2}} \div x^{-\frac{1}{3}} = x^{\frac{1}{2}-\left(-\frac{1}{3}\right)} = x^{\frac{1}{2}+\frac{1}{3}} = x^{\frac{5}{6}}$.

(iii) $\left(x^2\right)^{-3} = x^{2(-3)} = x^{-6}$.

(iv) $\left(\sqrt[3]{x}y\right)^3 = \left(\sqrt[3]{x}\right)^3 \left(y^1\right)^3 = \left(x^{\frac{1}{3}}\right)^3 \left(y^1\right)^3 = x^{\frac{1}{3}(3)} y^{1(3)} = xy^3$. **[5]**

(c) (i) $15.9950 = 16.00$ to two decimal places.

(ii) $0.0002345 = 0.00023$ to two significant figures.

(iii) $1530.13 = 1.53013 \times 10^3$ in scientific notation.

(iv) $0.0000205 = 2 \times 10^{-5}$ in scientific notation to one significant figure. **[4]**

(d) $(x^2 - x - 1) - (-x + 2) = x^2 + (-x + x) + (-1 - 2) = x^2 - 3$. **[1]**

(e)

$$\begin{aligned}(x^4 + 3x^2)(-x + 2) &= (x^4)(-x + 2) + (3x^2)(-x + 2)\\ &= (x^4)(-x) + (x^4)(2) + (3x^2)(-x) + (3x^2)(2)\\ &= -x^{4+1} + 2x^4 - 3x^{2+1} + 6x^2\\ &= -x^5 + 2x^4 - 3x^3 + 6x^2.\end{aligned}$$

[2]

(f)
$$
\begin{array}{r}
x - 1 \\
x + 2 \overline{) \; x^2 \quad + x + 1} \\
\underline{- x^2 - 2x \quad\quad\;\;} \\
- x + 1 \\
\underline{x + 2} \\
3
\end{array}
$$

This tells us that $\dfrac{x^2 + x + 2}{x + 2} = x - 1 + \dfrac{3}{x + 2}$.
So the quotient is $x - 1$ and the remainder is 3. **[4]**

(g) $\sum_{i=-1}^{2} i^3 = (-1)^3 + 0^3 + 1^3 + 2^3 = -1 + 0 + 1 + 8 = 8.$ **[2]**

(h) $\binom{8}{3} = \dfrac{8 \times 7 \times 6}{3 \times 2} = 56.$ **[2]**

(i)

$$
\begin{aligned}
(2x - y)^3 &= (2x)^3 + \binom{3}{1}(2x)^2(-y) + \binom{3}{2}(2x)(-y)^2 + (-y)^3 \\
&= 8x^3 - 12x^2y + 6xy^2 - y^3.
\end{aligned}
$$

[4]

2. (a) Here our line is parallel to a line that has slope -2, so our line also has slope $m = -2$.
Hence the equation of the line is $y = -2x + c$, where we still have to find c.
On substituting $x = 1$ and $y = -3$ into $y = -2x + c$, we obtain $-3 = -2(1) + c$, so that $c = -3 + 2(1) = -1$.
Hence the equation of the line is $y = -2x - 1$. **[2]**

(b) If we add the second equation to four times the first we obtain

$$
\begin{array}{rrcrcr}
 & 16x & + & -4y & = & 24 \\
+ & 3x & + & 4y & = & -5 \\
\hline
 & 19x & & & = & 19
\end{array}
$$

Hence $x = 1$ and on substituting this into the first equation we get
$4(1) - y = 6$, so that $y = 4 - 6 = -2$.
Thus the solution is $x = 1$ and $y = -2$. **[3]**

(c) Using $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (-1, 2)$, the formula tells us that the length of the line segment is

$$
\sqrt{(-1 - (-2))^2 + (2 - 1)^2} = \sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}.
$$

[1]

3. (a)

$$\begin{aligned}3x^2-2x-5 &= 3\left\{x^2-\frac{2}{3}x-\frac{5}{3}\right\}\\ &= 3\left\{\left(x-\frac{1}{3}\right)^2-\frac{1}{9}-\frac{5}{3}\right\}\\ &= 3\left\{\left(x-\frac{1}{3}\right)^2-\frac{16}{9}\right\}\\ &= 3\left(x-\frac{1}{3}\right)^2-\frac{16}{3}.\end{aligned}$$

[3]

(b) In this case $a=3$, $b=-2$ and $c=-5$.
Hence the solutions of the equation are

$$\begin{aligned}x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ &= \frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-5)}}{2(3)}\\ &= \frac{2\pm\sqrt{4+60}}{6}\\ &= \frac{2\pm\sqrt{64}}{6}\\ &= \frac{2\pm 8}{6}\\ &= -1 \text{ or } \frac{5}{3}.\end{aligned}$$

[2]

(c) From Part (b) we know that the graph cuts the x-axis when $x=-1$ and when $x=\frac{5}{3}$.
Next, when $x=0$, $y=-5$, so the graph cuts the y-axis when $y=-5$.
We also know the graph is U-shaped since $a>0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a},-\frac{b^2-4ac}{4a}\right)=\left(-\frac{-2}{2(3)},-\frac{(-2)^2-4(3)(-5)}{4(3)}\right)=\left(\frac{1}{3},-\frac{16}{3}\right).$$

We now have all the information we need and I have sketched the graph in Figure 1.

[4]

4. (a) (i) This is a function.
Its domain is $\mathbb{R}^-$ and its codomain is $\mathbb{R}^+$.

(ii) This is not a function.
For example, $f(0)$ is not defined, since -1 does not lie in the codomain. **[4]**

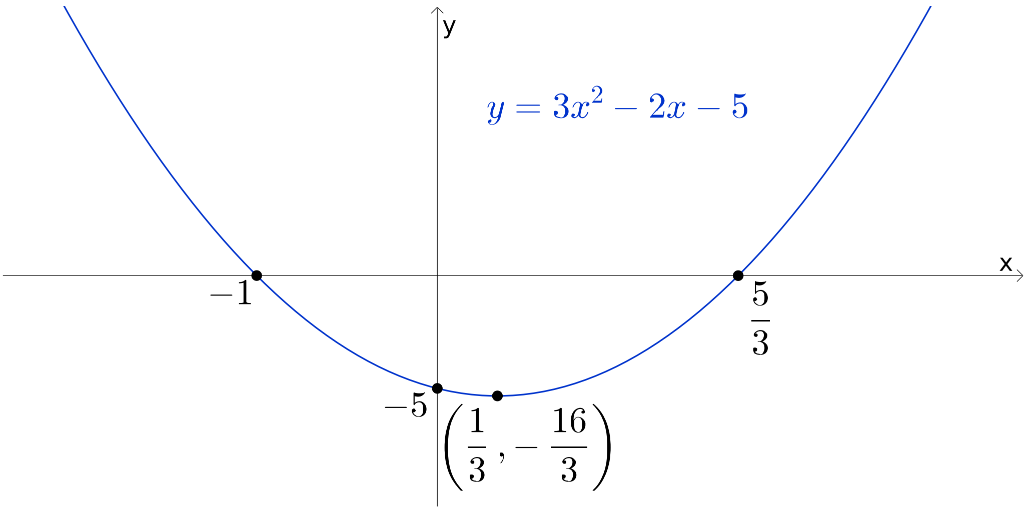

Figure 1: The Graph of the function $y = 3x^2 - 2x - 5$.

(b) Figure 2 shows the graph of the function

$$\begin{aligned} f\colon \{-2,-1,0,1,3\} &\to \{1,2,4\} \\ -2 &\mapsto 1 \\ -1 &\mapsto 4 \\ 0 &\mapsto 1 \\ 1 &\mapsto 4 \\ 3 &\mapsto 1 \end{aligned}$$

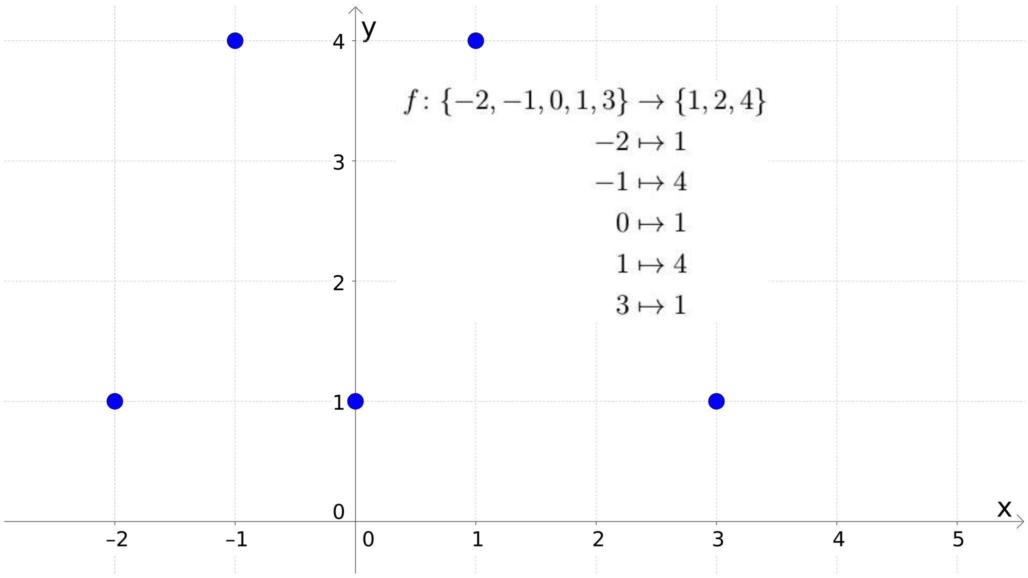

Figure 2: The graph of the function defined in Question 4(b).

[2]

(c) The function k crosses the x-axis so it must be a log function. In adition k increases as x increases, so it can't be (vi) and so must be (v).
Next g and l lie below the x-axis, so they must be (ii) and (iv). Now $y = 6^x$ increases as x increases, so $y = -6^x$ decreases as x increases. On the other hand $y = \left(\frac{2}{3}\right)^x$ decreases as x increases, so $y = -\left(\frac{2}{3}\right)^x$ increases as x increases. Thus g must be (iv) and l must be (ii).
Finally h lies above the x-axis, so must be (i) or (iii). However $y = 5^x$ increases as x increases, so it can't be h. Thus h must be (iii).
Summarizing: g is (iv), h is (iii), k is (v) and l is (ii). **[4]**

(d) (i) This function is not injective since $f(A) = 1 = f(D)$.
It is not surjective since there is no x with $f(x) = 2$.
It is not bijective since it is neither injective nor surjective.

(ii) This function is injective.
It is not surjective since there is no x with $f(x) = 0$.
It is not bijective since it is not surjective. **[3]**

(e) Neither of the functions in Part (d) are bijective, so neither of them have an inverse.

[1]

5. (a) $300^\circ = 300 \times \frac{\pi}{180} = \frac{5\pi}{3}$ Radians. **[1]**

(b) $\frac{7\pi}{12}$ Radians $= \left(\frac{7\pi}{12} \times \frac{180}{\pi}\right)^\circ = 105^\circ$. **[1]**

(c) In this case we want to find $\cos(\theta)$ when $\theta = \frac{7\pi}{6}$.

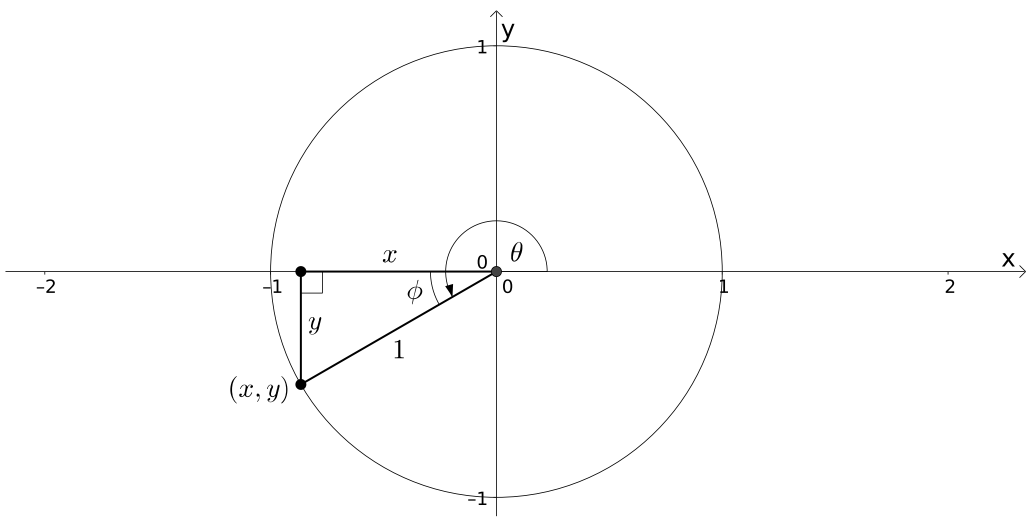

Figure 3: Calculation of $\cos\left(\frac{7\pi}{6}\right)$.

Looking at Figure 3, we see that we need to find x, since this is by definition $\cos\left(\frac{7\pi}{6}\right)$. Now, also from Figure 3, $\phi = \frac{7\pi}{6} - \pi = \frac{\pi}{6}$ (where we are just treating ϕ as an angle rather than a directed angle). Hence, using the table of common values, $\cos(\phi) = \frac{1}{2}$. But also by definition $\cos(\phi) = |x|$ (since the hypotenuse has length 1). Now, since x is negative, $x = -|x|$ and so $\cos\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$. **[4]**

(d) (i) Here we will first use $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$.

We have $\cos\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{2} - \frac{5\pi}{6}\right) = \sin\left(-\frac{\pi}{3}\right)$.

Next we will use $\sin(-\theta) = -\sin(\theta)$ and our table of common values to obtain $\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$. Hence $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

(ii) We will first use the fact that the sine function repeats every 2π.

Thus $\sin\left(\frac{7\pi}{4}\right) = \sin\left(\frac{7\pi}{4} - 2\pi\right) = \sin\left(-\frac{\pi}{4}\right)$.

We can now use our table of common values and $\sin(-\theta) = -\sin(\theta)$ to obtain $\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$. Hence $\sin\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$. **[4]**

(e) Using the sine rule in the form $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$, we obtain $\frac{a}{\sin(69°)} = \frac{6}{\sin(58°)}$.

Thus $a = \frac{6\sin(69°)}{\sin(58°)} \simeq 6.6051$. **[3]**

6. (a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x.
\end{aligned}$$

[3]

(b) (i) $f'(x) = 0$.

(ii) $f'(x) = 3x^{3-1} = 3x^2$.

(iii) $f'(x) = \frac{1}{x}$.

(iv) $f'(x) = -2(-\sin(-2x)) = 2\sin(-2x)$.

(v) $f'(x) = 2\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) - (-\cos(-x)) + 2\left(-4e^{-4x}\right) = x^{-\frac{1}{2}} + \cos(-x) - 8e^{-4x}$. **[6]**

7. (a) $\displaystyle\int 2\,dx = 2x + c.$ **[1]**

(b) $\displaystyle\int_1^2 x^5\,dx = \left[\frac{1}{6}x^6\right]_1^2 = \frac{1}{6}(2^6) - \frac{1}{6}(1^6) = \frac{63}{6}.$ **[2]**

(c)

$$\begin{aligned}\int_0^\pi \cos\left(\frac{1}{2}x\right)\,dx &= \left[\frac{1}{1/2}\sin\left(\frac{1}{2}x\right)\right]_0^\pi \\ &= \left[2\sin\left(\frac{1}{2}x\right)\right]_0^\pi \\ &= 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0) \\ &= 2 - 0 \\ &= 2.\end{aligned}$$

[2]

(d)

$$\begin{aligned}\int e^{2x} + x^{-\frac{3}{4}}\,dx &= \frac{1}{2}e^{2x} + \frac{1}{-\frac{3}{4}+1}x^{-\frac{3}{4}+1} + c \\ &= \frac{1}{2}e^{2x} + \frac{1}{1/4}x^{\frac{1}{4}} + c \\ &= \frac{1}{2}e^{2x} + 4x^{\frac{1}{4}} + c\end{aligned}$$

[2]

8. (a) (i) The mean is $\overline{x} = \frac{1}{7}(3 + 2 + (-8) + 8 + 2 + 5 + (-5)) = \frac{7}{7} = 1.$

(ii) The list in ascending order is $-8, -5, 2, 2, 3, 5, 8$.
Since there are seven numbers (an odd number), the median is
$m = x_{\frac{7+1}{2}} = x_4 = 2.$

(iii) There are 2 twos and one of each of the other numbers, so the mode is 2.

(iv) Since we have an odd number of numbers, we discard the median and split the remaining numbers into a lower half $-8, -5, 2$ and an upper half $3, 5, 8$. There are three numbers in each of these new groups (an odd number), so in each case the median is $x_{\frac{3+1}{2}} = x_2$. Thus the lower quartile is $Q_1 = -5$ and the upper quartile is $Q_3 = 5$. Hence the interquartile range is $Q_3 - Q_1 = 5 - (-5) = 10.$

[5]

(b) There are five points, so $n = 5$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{5} x_i = -4 + (-2) + 0 + 3 + 4 = 1$$
$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{5} y_i = -2 + (-2) + 0 + 1 + 2 = -1$$

$$\begin{aligned}\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{5} x_i y_i\\ &=(-4)(-2)+(-2)(-2)+(0)(0)+(3)(1)+(4)(2)\\ &=8+4+0+3+8\\ &=23.\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{5} x_i^2\\ &=(-4)^2+(-2)^2+0^2+3^2+4^2\\ &=16+4+0+9+16\\ &=45.\end{aligned}$$

Hence

$$\begin{aligned}m &= \frac{n\left(\sum_{i=1}^{n} x_i y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}\\ &= \frac{5(23)-(1)(-1)}{5(45)-1^2}\\ &= \frac{116}{224}\\ &= \frac{29}{56}\\ &\simeq 0.518,\end{aligned}$$

and

$$c = \overline{y} - m\overline{x} = \frac{\sum_{i=1}^{5} y_i}{5} - m\frac{\sum_{i=1}^{5} x_i}{5} = \frac{-1}{5} - \frac{29}{56}\times\frac{1}{5} = -\frac{17}{56} \simeq -0.304.$$

Thus the line of best fit is $y = \frac{29}{56}x - \frac{17}{56}$. [**8**]